Lecture 6: Recursion

Marvin Zhang
06/28/2016

Announcements

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- Alternate Exam Request: goo.gl/forms/FDQix4I5dNXPQDgw2

Roadmap

Introduction

Functions

Data

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Interpretation

Paradigms

Applications

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  - To understand the idea of *functional abstraction*
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  - recursion (today and tomorrow!)
  - orders of growth

Recursion

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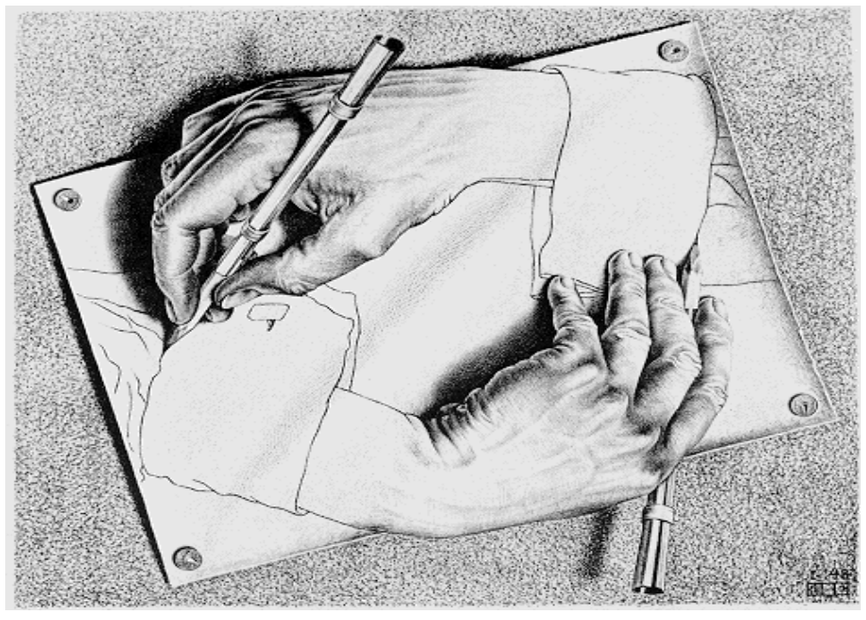
  - This implies that executing the body of a recursive function may require applying that function
- How is this possible? We'll see some examples next.



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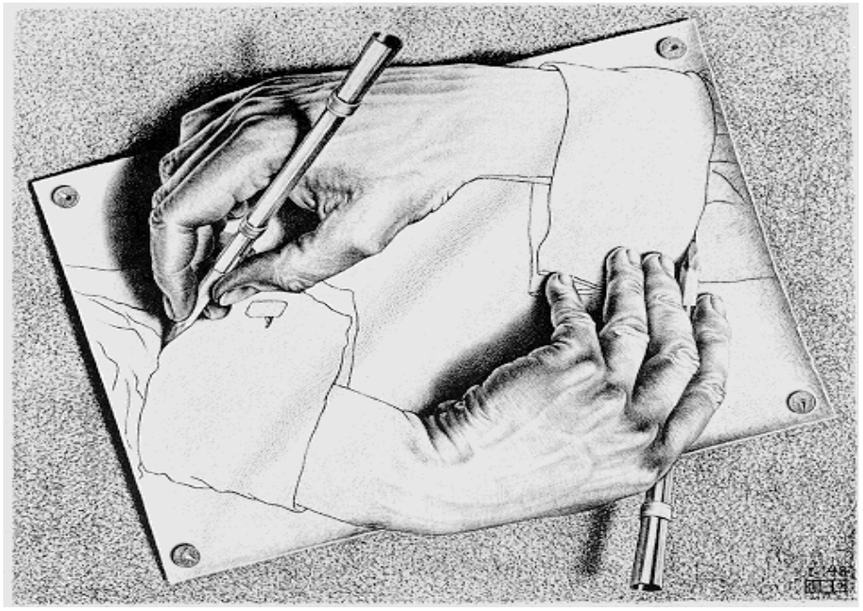

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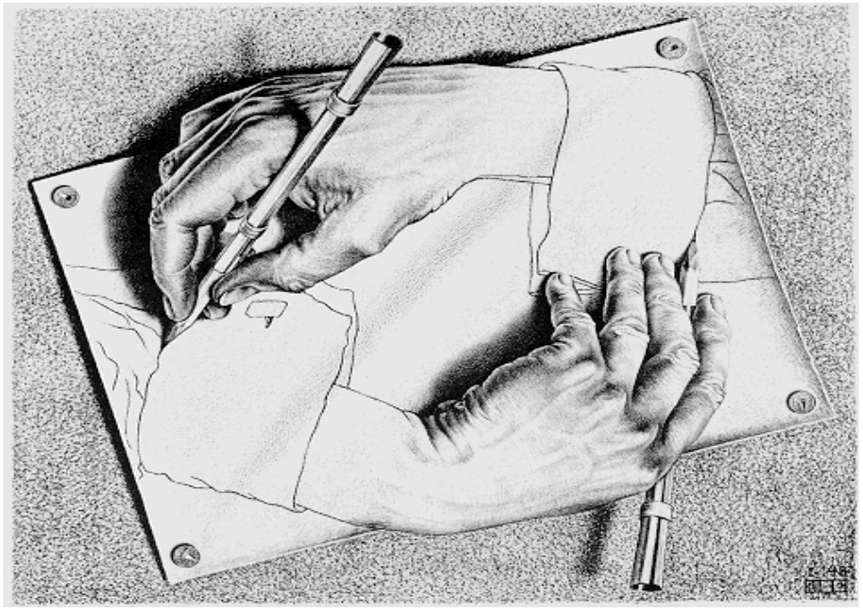

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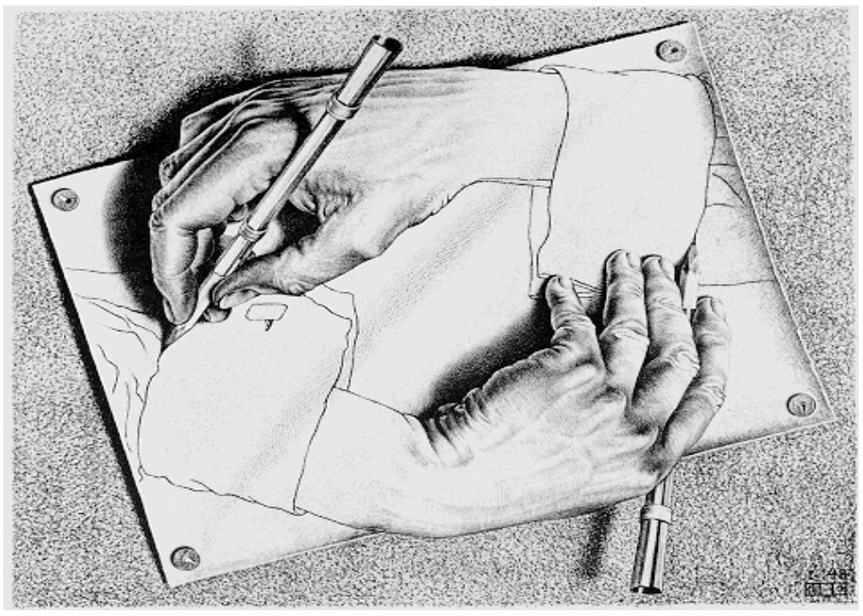

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Anatomy of a Recursive Function

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Verifying Correctness

The easy way, and the right way

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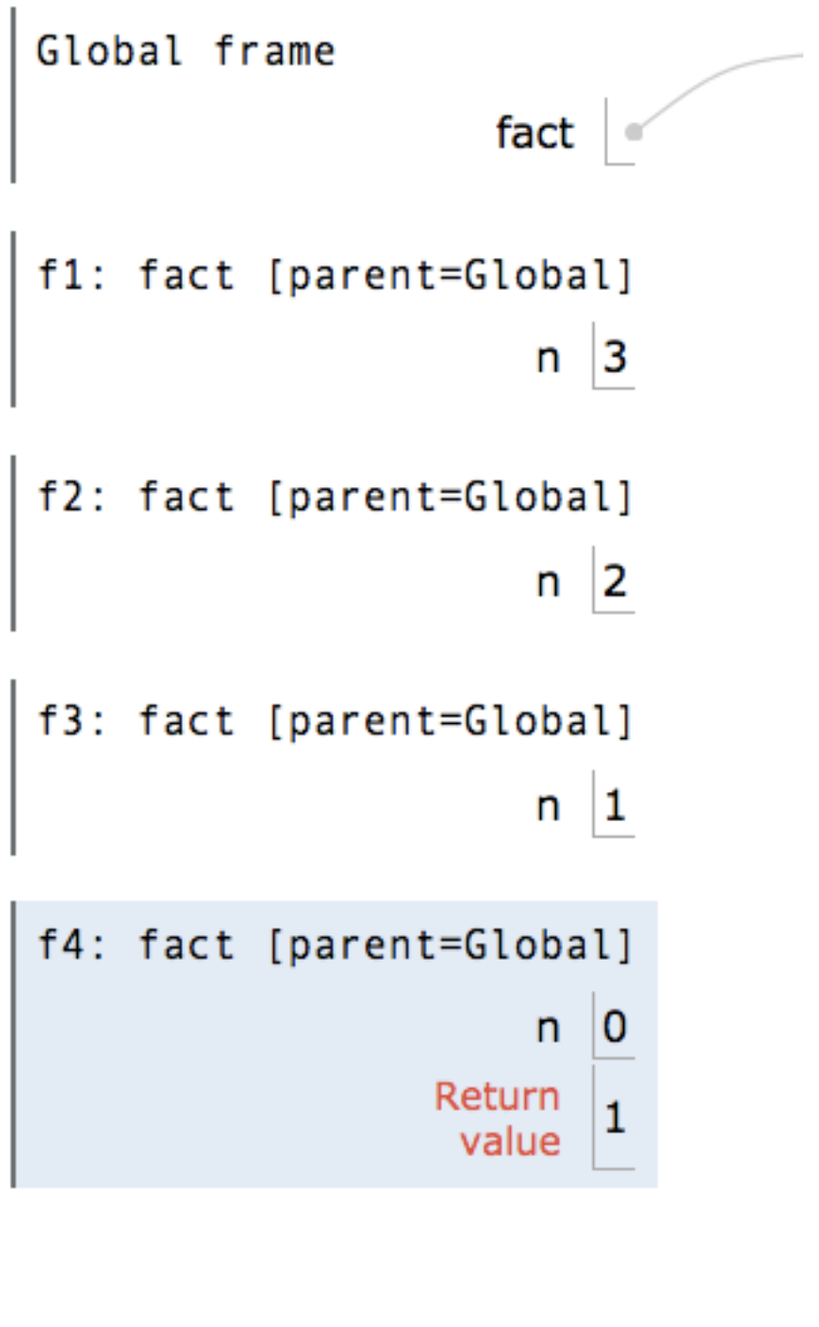

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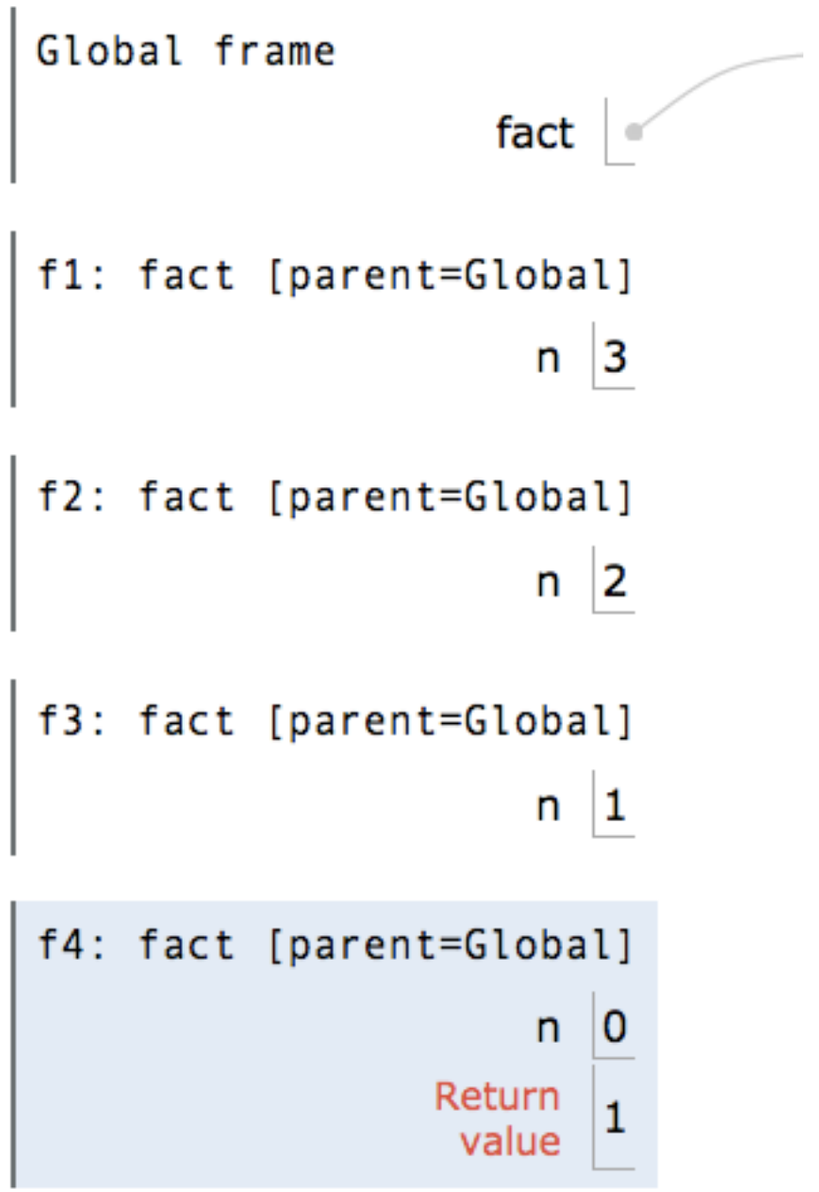

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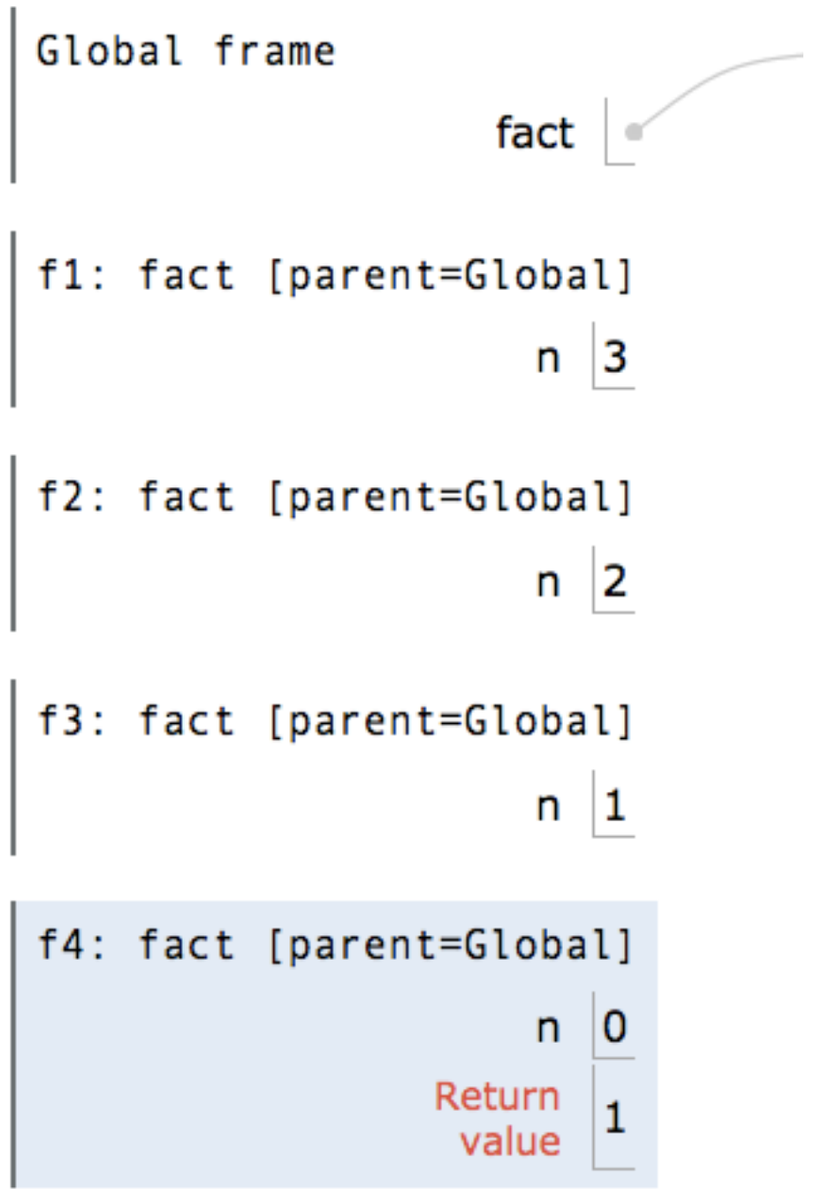

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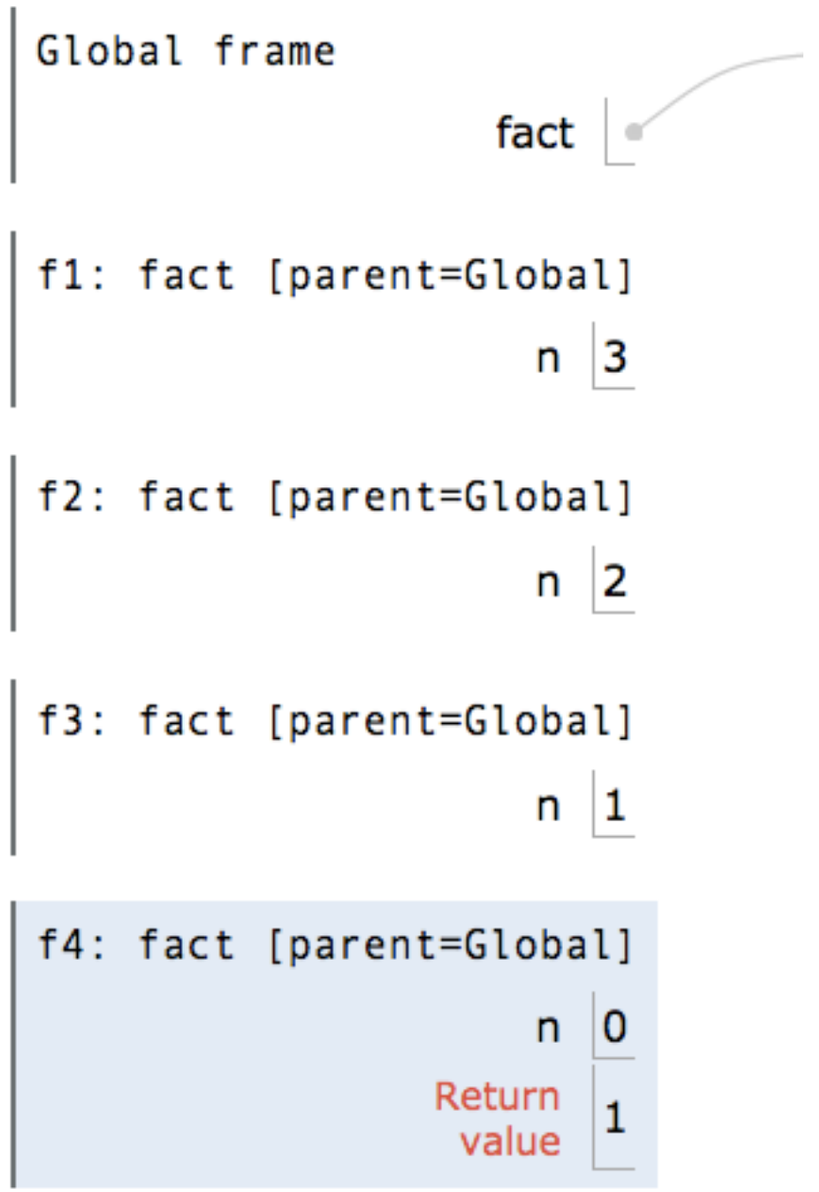

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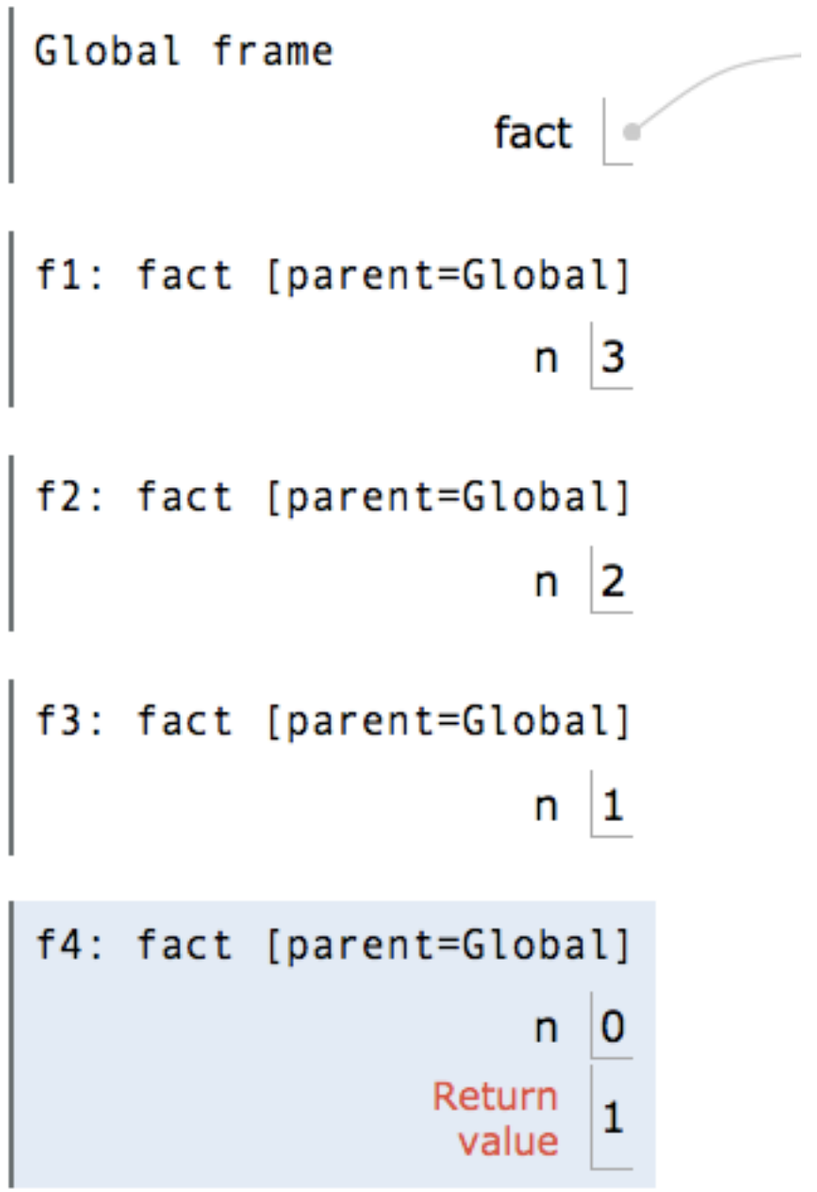

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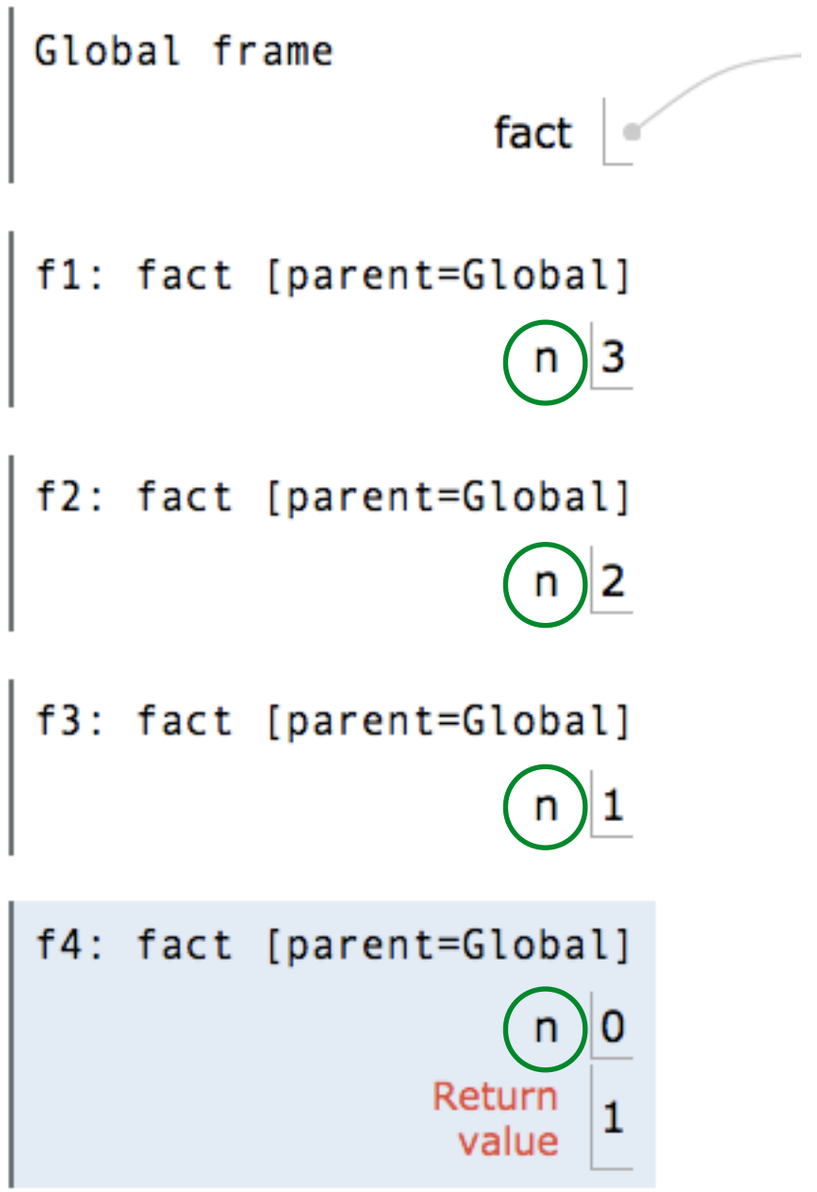

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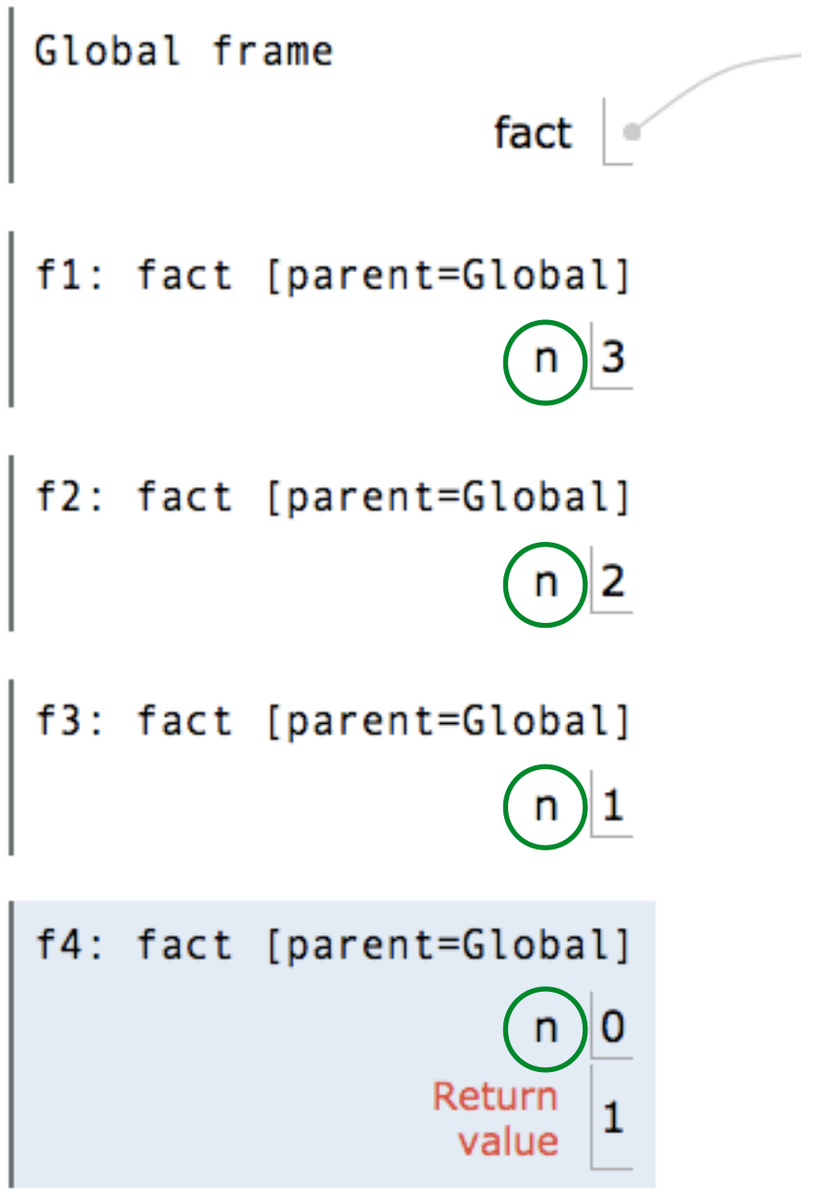
- What `n` evaluates to depends upon the current environment
- Each call to `fact` solves a simpler problem than the last: smaller `n`



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if n < 0:
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```
if n == 1:
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if n < 10:
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if n < 100:
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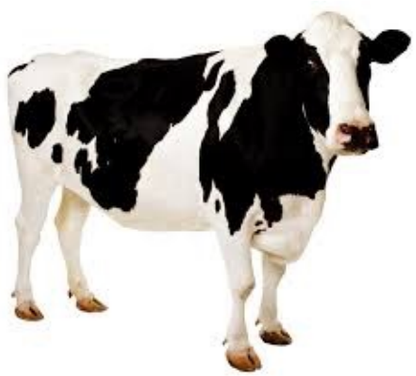

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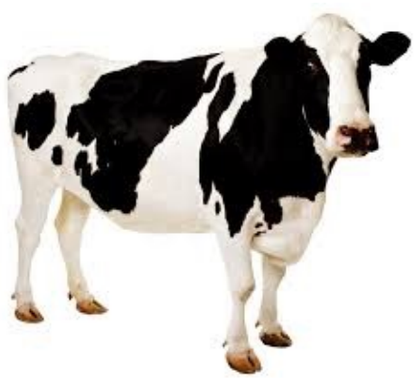

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Iteration vs Recursion

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    while k <= n:
        total, k = total*k, k+1
    return total
```

Using recursion:

```
def fact(n):
    if n == 0:
        return 1
    else:
        return n * fact(n-1)
```

Math: $n! = \prod_{k=1}^{n} k$

$$n! = \begin{cases} 1 & \text{if } n = 0 \\ n \cdot (n-1)! & \text{otherwise} \end{cases}$$

Names: `n, total, k, fact_iter`

`n, fact`

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```
def reverse(word):
    """Return the reverse of the string word."""
    if len(word) < 2:
        return word
    else:
        return reverse(word[1:]) + word[0]
```

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